

# ON GROWTH OF RANDOM GROUPS OF INTERMEDIATE GROWTH

MUSTAFA G. BENLI, ROSTISLAV GRIGORCHUK AND YAROSLAV VOROBETS

**ABSTRACT.** We study the growth of typical groups from the family of  $p$ -groups of intermediate growth constructed by the second author. We find that, in the sense of category, a generic group exhibits oscillating growth with no universal upper bound. At the same time, from a measure-theoretic point of view (i.e., almost surely relative to an appropriately chosen probability measure), the growth function is bounded by  $e^{n^\alpha}$  for some  $\alpha < 1$ .

## 1. INTRODUCTION

There are few approaches to randomness in group theory. The most known are associated with the names of Gromov and Olshanskii. For the account of these approaches and further literature see, [Oll05]. As far as the authors are concerned, these approaches deal with the models when in certain classes of finitely presented groups one locates "generic" finite presentations with prescribed properties. These models are variation of the *density model* and randomness in them appears in the form of *frequency* or *density*, which correspond to the classic "naive" approach to probability in mathematics.

The modern Kolmogorov's approach to probability assumes existence of a space supplied with a sigma-algebra of measurable sets and a probability measure on it.

There can be different constructions of spaces of groups and one of them was suggested in [Gri84], where Gromov's idea from [Gro81] of convergence of marked metric spaces was transformed into the notion of the compact totally disconnected topological metrizable space  $\mathcal{M}_k$  of marked  $k$ -generated groups,  $k \geq 2$ . Later it was discovered that this topology is related to the Chabauty topology in the space of normal subgroups of the free group of rank  $k$  [Cha00]. Observe that in general the Chabauty topology is defined in the space of closed subgroups of a locally compact group (there is analogous notion in differential geometry [BP92]), and that in the case of a discrete group  $G$  it is nothing but the topology induced on the set of subgroups of  $G$  by the Tychonoff topology of the space  $\{0, 1\}^G$ .

The main result of [Gri84] is the construction of the first examples of groups of intermediate growth, thus answering a question of Milnor [Mil68]. In fact, an uncountable family of 3-generated groups was introduced and studied in [Gri84], and among other results it was shown there that the set of rates of growth of

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finitely generated groups has the cardinality of the continuum, and that there are pairs of groups with incomparable growth (the growth rates of two groups are different but neither grows faster than the other; in fact, the space of rates of growth of finitely generated groups contains an anti-chain of the cardinality of the continuum). The possibility of such phenomenon is based on the fact that there are groups with *oscillating growth*, i.e. groups whose growth on different parts of the range of the argument of growth function (which is a set of natural numbers) behaves alternatively in two fashions: in the intermediate (between polynomial and exponential) way and exponential way. In this paper we will use one particular form of oscillating property which will be defined below.

The construction in [Gri84] deals with torsion 2-groups of intermediate growth. It also provides interesting examples of self-similar groups and first examples of just-infinite branch groups (see [GŠ07]). Later a similar construction of  $p$ -groups of intermediate growth was produced for arbitrary prime  $p$  as well as the first example of a torsion free group of intermediate growth [Gri85a].

The reason for introducing the space  $\mathcal{M}_k$  of marked groups in [Gri84] was to show that this space in the cases  $k = 2, 3$  (and hence for all  $k$ ) contains a closed subset of groups homeomorphic to a Cantor set consisting primarily of torsion groups of intermediate growth. Later other interesting families of groups constituting a Cantor set of groups and satisfying various properties were produced and used for answering different questions [Cha00, Nek07]. The topology in the space  $\mathcal{M}_k$  was used in [Gri84, Gri85b] not only for study of growth but also for investigations of algebraic properties of the involved groups. For instance, among many ways of showing that the involved groups are not finitely presented, there is one which makes use of this topology (the topic of finite presentability of groups in the context of growth, amenability and topology is discussed in detail in [BGH12]). At present there is a big account of results related to the space of marked groups and various algebraic, geometric and asymptotic properties of groups including such properties as (T)-property of Kazhdan, local embeddability into finite or amenable groups (so-called LEF and LEA properties), being sofic and various other properties (see [CSC10] for a comprehensive source of these).

For each  $k \geq 2$  there is a natural embedding of  $\mathcal{M}_k$  into  $\mathcal{M}_{k+1}$  and one can consider the inductive limit  $\mathcal{M} = \lim_k \mathcal{M}_k$  which is a locally compact totally disconnected space. As was observed by Champetier in [Cha00] the group of Nielsen transformations over infinite generating set acts naturally on this space with orbits consisting of isomorphic groups. Any Baire measure on  $\mathcal{M}$ , i.e., a measure defined on the sigma algebra generated by compact  $G_\delta$  sets (countable intersections of open sets) with finite values on compact sets, that is invariant (or at least quasi-invariant) with respect to this action would be a good choice for the model of random finitely generated group (this approach based on discussions of the second author with E. Ghys is presented in [Gri05]). Unfortunately, at the moment no such measures were produced. This is also related to the question of existence of "good" measures invariant (or quasi-invariant) under the action of the

automorphism group of a free group  $F_k$  of rank  $k \geq 2$ , with support in the set of normal subgroups of  $F_k$ .

Fortunately, another approach can be used. It is based on the following idea. Assume we have a compact  $X \subset \mathcal{M}_k$  of groups and a continuous map  $\tau : X \rightarrow X$ . Then by the Bogolyubov-Krylov Theorem there is at least one  $\tau$ -invariant probability measure  $\mu$  on  $X$ . Suppose also that we have a certain group property  $\mathcal{P}$  (or a family of properties), and that the subset  $X_{\mathcal{P}} \subset X$  of groups satisfying this property is measurable  $\tau$ -invariant (i.e.  $\tau^{-1}(X_{\mathcal{P}}) = X_{\mathcal{P}}$ ). Then one may be interested in the measure  $\mu(X_{\mathcal{P}})$  which is 0 or 1 in the case of ergodic measure (i.e. when the only invariant measurable subsets up to sets of  $\mu$  measure 0 are empty set and the whole set  $X$ ). Observe that by (another) Bogolyubov-Krylov theorem, ergodic measure always exist in the situation of a continuous map on a metrizable compact space and are just extreme points of the simplex of invariant measures. The described model allows to speak about typical properties of a random group from the family  $(X, \mu)$ .

The alternative approach when the measure  $\mu$  is not specified is the study of the typical properties of groups in compact  $X$  from topological (or categorical) point of view i.e., in the sense of Baire category. Under this approach a group property  $\mathcal{P}$  is typical if the subset  $X_{\mathcal{P}}$  is co-meager i.e. its complement  $X \setminus X_{\mathcal{P}}$  is meager (a countable union of nowhere dense subsets of  $X$ ). It happens quite often that what is typical in the measure sense is not typical in the sense of category and this paper gives one more example of this sort.

## 2. STATEMENT OF MAIN RESULTS

In Ergodic Theory (and more generally in Probability Theory), one of the most important models is the model of a shift in a space of sequences. Given a finite alphabet  $Y = \{s_1, \dots, s_k\}$ , one considers a space  $\Omega = Y^{\mathbb{N}}$  of infinite sequences  $\omega = (\omega_n)_{n=1}^{\infty}$ ,  $\omega_n \in Y$  endowed with the Tychonoff product topology. A natural transformation in such space is a shift  $\tau : \Omega \rightarrow \Omega$ ,  $(\tau(\omega))_n = \omega_{n+1}$ . There is a lot of invariant measures for the dynamical system  $(\Omega, \tau)$  and in fact the simplex  $M_{\tau}(\Omega)$  of invariant measures is Poulsen simplex (i.e. ergodic measures are dense in weak-\* topology).

Let  $p$  be prime and consider the set  $\{0, 1, \dots, p\}$  as an alphabet with the corresponding set  $\Omega_p = \{0, 1, \dots, p\}^{\mathbb{N}}$  of infinite sequences endowed with the shift  $\tau : \Omega_p \rightarrow \Omega_p$ . Let  $\Omega_{p,0}$  denote the subset of sequences which are eventually constant and  $\Omega_{p,\infty}$  the set of sequences in which all symbols  $\{0, 1, \dots, p\}$  appear infinitely often. Note that  $\Omega_{p,\infty}$  and  $\Omega_{p,0}$  are  $\tau$  invariant.

In [Gri84, Gri85a] for each  $\omega \in \Omega_p$  a group  $G_{\omega}$  with a set  $S_{\omega} = \{a, b_{\omega}, c_{\omega}\}$  of three generators acting on the interval  $[0, 1]$  by Lebesgue measure preserving transformations was constructed. One of the specific features of this construction is that if two sequences  $\omega, \eta \in \Omega_p$ , which are not eventually constant, have the same prefix of length  $n$  then the corresponding groups  $G_{\omega}, G_{\eta}$  have isomorphic Cayley

graphs in the neighborhood of the identity element of radius  $2^{n-1}$ . Replacing the groups  $G_\omega, \omega \in \Omega_{p,0}$  with the appropriate limits (again denoted by  $G_\omega$ ), that is, taking the closure of the set  $\{(G_\omega, S_\omega) \mid \omega \in \Omega_p \setminus \Omega_{p,0}\}$  in  $\mathcal{M}_3$ , one obtains a compact subset  $\mathcal{G}_p = \{(G_\omega, S_\omega) \mid \omega \in \Omega_p\}$  of  $\mathcal{M}_3$  which is homeomorphic to  $\Omega_p$  (via the correspondence  $\omega \mapsto (G_\omega, S_\omega)$ ) and hence homeomorphic to a Cantor set. In what follows we will continue to keep the notation  $G_\omega, \omega \in \Omega_p$  to denote this groups after this modification. Also quite often we will identify  $\mathcal{G}_p$  with  $\Omega_p$ . In the case  $p = 2$  the new limit groups  $G_\omega, \omega \in \Omega_{2,0}$  are known to be virtually metabelian groups of exponential growth and there is no analogous result for the case  $p > 2$ . This is the underlying reason that Theorem 3 below is stated for only  $p = 2$ .

Another important feature of the construction is that for all but countably many  $\omega \in \Omega_2$  and for all  $\omega \in \Omega_p$  the groups  $G_\omega$  and  $(G_{\tau(\omega)})^p$  (direct product of  $p$  copies of  $G_\omega$ ) are abstractly commensurable (i.e., contain isomorphic subgroups of finite index). Thus shift  $\tau$  preserves many of group properties on the set of full measure when  $\mu$  is a  $\tau$  invariant measure supported on  $\Omega_{p,\infty}$ , for instance, the property to be a torsion group. While for some properties of the groups  $G_\omega$  it is quite easy to decide whether it is typical or not, there are some properties for which such a question is more difficult to answer. Among them is the property to have a growth function bounded from above (or below) by a specific function.

Given functions  $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{N}$ , we write  $f_1 \preceq f_2$  if  $f_1$  grows no faster than  $f_2$  and  $f_1 \sim f_2$  if  $f_1 \preceq f_2$  and  $f_2 \preceq f_1$ .  $f_1 \prec f_2$  means that  $f_1 \preceq f_2$  but  $f_1 \not\sim f_2$  (precise definitions are given in Section 3). For any  $\omega \in \Omega_p$  let  $\gamma_\omega(n)$  denote the growth function of the group  $G_\omega$ . It was shown in [Gri84, Gri85a] that if symbols of the alphabet  $\{0, 1, \dots, p\}$  are more or less uniformly distributed in a sequence  $\omega$  then  $\gamma_\omega(n)$  grows slower than  $e^{n^\alpha}$  with constant  $\alpha < 1$ . At the same time, in the case  $p = 2$  for any function  $f(n) \prec e^n$  there is a sequence  $\omega$  such that  $\gamma_\omega(n)$  grows not slower than  $f(n)$ .

The upper and lower bounds by functions of the type  $e^{n^\alpha}$  with constant  $0 < \alpha < 1$  are of special importance in study of growth of finitely generated groups and there is a number of interesting results and conjectures associated with them. One of the main conjectures says that if the growth of a group  $G$  is slower than  $e^{\sqrt{n}}$  then it is actually polynomial [Gri91, Gri11, Gri12].

We are ready to formulate our results.

**Theorem 1.** *Suppose  $\mu$  is a Borel probability measure on  $\Omega_p$  that is invariant and ergodic relative to the shift transformation  $\tau : \Omega_p \rightarrow \Omega_p$ .*

- a) *If the measure  $\mu$  is supported on  $\Omega_{p,\infty}$ , then there exists  $\alpha = \alpha(\mu, p) < 1$  such that  $\gamma_\omega(n) \preceq e^{n^\alpha}$  for  $\mu$ -almost all  $\omega \in \Omega_p$ .*
- b) *In the case  $\mu$  is the uniform Bernoulli measure on  $\Omega_2$ , one can take  $\alpha = 0.999$ .*

If  $T : \mathcal{G}_p \rightarrow \mathcal{G}_p$  is a map induced by the shift  $\tau$ , our result can be interpreted as follows: For any "reasonable"  $T$ -invariant measure  $\mu$  on  $\mathcal{G}_p \subset \mathcal{M}_3$ , a typical group in  $\mathcal{G}_p$  has growth bounded by  $e^{n^\alpha}$ , where  $\alpha = \alpha(\mu, p) < 1$ .

The bound for  $\alpha$  given in part (b) of the Theorem 1 is far from to be optimal, but getting an essentially better bound would require more work. In any case it can not be below  $1/2$  as for all groups  $G_\omega$  of intermediate growth the corresponding growth function is bounded from below by  $e^{n^{1/2}}$  [Gri84, Gri85a, Gri89]. The gap conjecture, discussed in [Gri12] and proven in certain cases, gives more information about what one can expect concerning possible optimal values of  $\alpha$ .

In fact, there is nothing special about the space  $\mathcal{M}_3$  and the following holds:

**Theorem 1'.** *For any  $k \geq 2$  and prime  $p$ ,  $\mathcal{M}_k$  contains a compact subset  $\mathcal{K}_k = \{(M_\omega, L_\omega) \mid \omega \in \Omega_p\}$  homeomorphic to  $\Omega_p$  (via the map  $\omega \mapsto (M_\omega, L_\omega)$ ) such that if  $\mu$  is a measure supported on  $\Omega_{p,\infty}$  there exists  $\alpha = \alpha(\mu, p) < 1$  such that  $\gamma_{M_\omega}(n) \preceq e^{n^\alpha}$  for  $\mu$ -almost all  $\omega \in \Omega_p$ .*

For  $k \geq 3$ , the group  $M_\omega$  is the same as  $G_\omega$ , with an appropriate generating set  $L_\omega$  of size  $k$ . For  $k = 2$ ,  $M_\omega$  is a 2-generated group constructed from  $G_\omega$  using an idea from [Gri84] (see Section 6).

The proof of Theorem 1 is based on the next result, which has its own interest. It improves [Gri85a, Theorem 3] (a generalization of [Gri84, Theorem 3.2]), which states that if the sequence  $\omega$  is regularly packed by symbols  $0, 1, 2$ , namely, if there exists  $k = k(\omega)$  such that each subsequence of length  $k$  of  $\omega$  contains all symbols  $\{0, 1, \dots, p\}$ , then there is  $\alpha < 1$  such that  $\gamma_\omega(n) \preceq e^{n^\alpha}$ .

To every infinite word  $\omega = l_1 l_2 \dots$  in  $\Omega_{p,\infty}$  we associate an increasing sequence of integers  $t_i = t_i(\omega)$ ,  $i = 0, 1, 2, \dots$ . Namely,  $t_i$  is the smallest integer such that the finite word  $l_1 l_2 \dots l_{t_i}$  can be split into  $i$  subwords each containing all letters  $\{0, 1, \dots, p\}$ . For any  $C \geq p + 1$  let  $\Omega_{p,C}$  denote the set of all infinite words  $\omega \in \Omega_{p,\infty}$  such that  $t_n(\omega) \leq Cn$  for sufficiently large  $n$ . Given  $\epsilon > 0$ , let  $\Omega_{p,C,\epsilon}$  denote the set of all  $\omega \in \Omega_{p,C}$  such that  $t_{n+1}(\omega) - t_n(\omega) \leq \epsilon t_n(\omega)$  for sufficiently large  $n$ .

**Theorem 2.** *Given  $C \geq p + 1$ , there exist  $\epsilon > 0$  and  $0 < \alpha < 1$  such that  $\gamma_\omega(n) \preceq e^{n^\alpha}$  for any  $\omega \in \Omega_{p,C,\epsilon}$ .*

Our next result deals exclusively with the case  $p = 2$  (the reason was explained earlier). Given two functions  $\gamma_1, \gamma_2 : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\gamma_1(n) \prec \gamma_2(n) \prec e^n$ , let us say that a group  $G$  has *oscillating growth of type*  $(\gamma_1, \gamma_2)$  if  $\gamma_1 \not\preceq \gamma_G$  and  $\gamma_G \not\preceq \gamma_2$ . The existence of groups with oscillating growth follows from the results of [Gri84]. The results of [KP11] and of [BE12], [Bri11] provide much more information in this direction.

Let  $\theta_0 = \log(2)/\log(2/x_0)$ , where  $x_0$  is the real root of the polynomial  $x^3 + x^2 + x - 2$ . We have  $\theta_0 < 0.767429$ .

**Theorem 3.**

- a) *For any  $\theta > \theta_0$  and any function  $f$  satisfying  $e^{n^\theta} \prec f(n) \prec e^n$ , there exists a dense  $G_\delta$  subset  $\mathcal{Z} \subset \mathcal{G}_2$  such that any group in  $\mathcal{Z}$  has oscillating growth of type  $(e^{n^\theta}, f)$ .*

- b) *There exists a dense  $G_\delta$  subset of  $\mathcal{G}_2$  which consists of groups with oscillating growth of type  $(e^{n^\theta}, e^{n^\beta})$  for every  $\theta$  and  $\beta$ ,  $\theta_0 < \theta < \beta < 1$ .*
- c) *Given any  $\epsilon > 0$  and function  $f$  satisfying  $\exp\left(\frac{n}{\log^{1-\epsilon} n}\right) \prec f(n) \prec e^n$ , there is a dense  $G_\delta$  subset  $\mathcal{E} \subset \{(G_\omega, S_\omega) \mid \omega \in \{0, 1\}^\mathbb{N}\}$  such that any group in  $\mathcal{E}$  has oscillating growth of type  $(\exp\left(\frac{n}{\log^{1-\epsilon} n}\right), f)$ .*

Again, all these results generalize to arbitrary  $k \geq 2$ . In particular, the following theorem holds.

**Theorem 3'.** *For each  $k \geq 2$ ,  $\theta > \theta_0$  and function  $f$  satisfying  $e^{n^\theta} \prec f(n) \prec e^n$ ,  $\mathcal{M}_k$  contains a compact subset  $\mathcal{C}_k$  homeomorphic to  $\Omega_2$ , such that there is a dense  $G_\delta$  subset  $\mathcal{C}'_k \subset \mathcal{C}_k$  which consists of groups with oscillating growth of type  $(e^{n^\theta}, f)$ .*

The reason why oscillating groups are typical in the categorical sense is the existence of a countable dense subset in  $\mathcal{G}_2$  consisting of (virtually metabelian) groups of exponential growth and a dense subset of groups with the growth equivalent to the growth of the *first Grigorchuk group*  $G_{(012)^\infty}$ , which is bounded by  $e^{n^{\theta_0}}$  due to a result of Bartholdi [Bar98] (see also [MP01]). Note also that this is the smallest upper bound known for any group of intermediate growth. It is used to prove part a). To prove part c), we use instead a result of Erschler [Ers04] stating that the growth of the group  $G_\omega$  for  $\omega = (01)^\infty \in \Omega_2$  is slower than  $\exp\left(\frac{n}{\log^{1-\epsilon} n}\right)$  for all  $\epsilon > 0$ .

Notice that the categorical approach for study of amenability of groups from the family  $\mathcal{G}_2$  was suggested by Stepin in [Ste96] where the fact that this family contains a dense set of virtually metabelian (and hence amenable) groups was used to show that amenability is a typical property of this family. In fact, all groups in  $\mathcal{G}_2$  are amenable as was shown in [Gri84], but Stepin's paper provided for the first time a categorical approach to study typical groups in compact subsets of the space of marked groups.

### 3. PRELIMINARIES

**Definition of the groups.** The original definition of groups in [Gri84, Gri85a] is in terms of measure preserving transformations of the unit interval. We will give here the alternative definition in terms of automorphisms of rooted trees. For the sake of notation we will focus on the case  $p = 2$  and the construction in the case  $p \geq 3$  is analogous. For more detailed account of this construction see [Gri99, Gri05].

Let us recall some notation:  $\Omega_2$  denotes the set all infinite sequences over the alphabet  $\{0, 1, 2\}$ . We identify  $\Omega_2$  with the product  $\{0, 1, 2\}^\mathbb{N}$  and endow it with the product topology. Let  $\Omega_{2,0}$  be the set of eventually constant sequences and  $\Omega_{2,\infty}$  be the set of sequences in which each letter 0,1,2 appears infinitely often.



Our notation here is different from [Gri84, Gri85a]. Let  $\tau : \Omega_2 \rightarrow \Omega_2$  denote the shift transformation, that is if  $\omega = l_1 l_2 \dots$  then  $\tau(\omega) = l_2 l_3 \dots$ . Note that both  $\Omega_{2,0}$  and  $\Omega_{2,\infty}$  are  $\tau$  invariant.

For each  $\omega \in \Omega_2$  we will define a subgroup  $G_\omega$  of  $Aut(\mathcal{T}_2)$ , where the latter denotes the automorphism group of the binary rooted tree  $\mathcal{T}_2$  whose vertices are identified with the set of finite sequences  $\{0, 1\}^*$ . Each group  $G_\omega$  is the subgroup generated by the four automorphisms denoted by  $a, b_\omega, c_\omega, d_\omega$  whose actions onto the tree is as follows:

For  $v \in \{0, 1\}^*$

$$a(0v) = 1v \text{ and } a(1v) = 0v$$

$$\begin{aligned} b_\omega(0v) &= 0\beta(\omega_1)(v) & c_\omega(0v) &= 0\zeta(\omega_1)(v) & d_\omega(0v) &= 0\delta(\omega_1)(v) \\ b_\omega(1v) &= 1b_{\tau(\omega)}(v) & c_\omega(1v) &= 1c_{\tau(\omega)}(v) & d_\omega(1v) &= 1d_{\tau(\omega)}(v), \end{aligned}$$

where

$$\begin{aligned} \beta(0) &= a & \beta(1) &= a & \beta(2) &= e \\ \zeta(0) &= a & \zeta(1) &= e & \zeta(2) &= a \\ \delta(0) &= e & \delta(1) &= a & \delta(2) &= a \end{aligned}$$

and  $e$  denotes the identity. From the definition, the following relations are immediate:

$$(1) \quad a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = e.$$

Observe that the group  $G_\omega$  is in fact 3-generated as one of generators  $b_\omega, c_\omega, d_\omega$  can be deleted from the generating set. We will use the notation  $A_\omega = \{a, b_\omega, c_\omega, d_\omega\}$  for this generating set while  $S_\omega$  will denote the reduced generating set  $\{a, b_\omega, c_\omega\}$  (as in Section 2). Algebraically, the action defines an embedding into the semi-direct product

$$\begin{aligned} \varphi_\omega : G_\omega &\rightarrow S_2 \quad \ltimes \quad (G_{\tau(\omega)} \times G_{\tau(\omega)}) \\ a &\mapsto (01) & (e &, e) \\ b_\omega &\mapsto & (\beta(\omega_1) &, b_{\tau(\omega)}) \\ c_\omega &\mapsto & (\zeta(\omega_1) &, c_{\tau(\omega)}) \\ d_\omega &\mapsto & (\delta(\omega_1) &, d_{\tau(\omega)}) \end{aligned}$$

where  $S_2$  is the symmetric group of order 2 and  $(01)$  denotes its non-identity element.

Given  $g \in G_\omega$  and  $x \in \{0, 1\}$  let us denote the  $x$  coordinate of  $\varphi_\omega(g)$  by  $g_x$  (or by  $g|_x$  to avoid possible confusion) so that  $\varphi_\omega(g) = \sigma_g(g_0, g_1)$ . Let us also extend this to all  $\{0, 1\}^*$  by

$$g_{xv} = (g_x)_v$$

where  $x \in \{0, 1\}$  and  $v \in \{0, 1\}^*$ . For  $g \in G_\omega$  and  $v \in \{0, 1\}^*$  the automorphism  $g_v$  will be called *the section of  $g$  at vertex  $v$* . Note that if  $v$  has length  $n$  and

$g \in G_\omega$ , then  $g_v$  is an element of  $G_{\tau^n(\omega)}$ . Given  $g, h \in G_\omega$  and  $v \in \{0, 1\}^*$  we have

$$(2) \quad (gh)_v = g_{h(v)}h_v$$

**Topology in the space of marked groups.** A *marked  $k$ -generated group* is a pair  $(G, S)$  where  $G$  is a group and  $S = \{s_1, \dots, s_k\}$  is an ordered set of (not necessarily distinct) generators of  $G$ . The *canonical map* between two marked  $k$ -generated groups  $(G, S)$  and  $(H, T)$  is the map that sends  $s_i$  to  $t_i$  for  $i = 1, 2, \dots, k$ . Let  $\mathcal{M}_k$  denote the *space of marked  $k$ -generated groups* consisting of marked  $k$ -generated groups where two marked groups are identified whenever the canonical map between them extends to an isomorphism of the groups.

There is a natural metric on  $\mathcal{M}_k$ : Two marked groups  $(G, S), (H, T)$  are of distance  $\frac{1}{2^m}$  where  $m$  is the largest natural number such that the canonical map between  $(G, S)$  and  $(H, T)$  extends to an isomorphism (of labeled graphs) from the ball of radius  $m$  (around the identity) in the Cayley graph of  $(G, S)$  onto the ball of radius  $m$  in the Cayley graph of  $(H, T)$ . This makes  $\mathcal{M}_k$  into a compact, totally disconnected topological space.

Alternatively, let  $F_k$  be free over the ordered basis  $X = \{x_1, \dots, x_k\}$  and let  $\mathcal{N}(F_k)$  denote the set of normal subgroups of  $F_k$ .  $\mathcal{N}(F_k)$  has a natural topology inherited from the space  $\{0, 1\}^{F_k}$  of all subsets of  $F_k$ .  $\mathcal{M}_k$  can be identified with  $\mathcal{N}(F_k)$  in the following way: Each  $(G, S) \in \mathcal{M}_k$  is identified with the kernel of the canonical map between  $(F_k, X)$  and  $(G, S)$ . Conversely each  $N \triangleleft F_k$  is identified with  $(F_k/N, \{\bar{x}_1, \dots, \bar{x}_k\})$  where  $\{\bar{x}_1, \dots, \bar{x}_k\}$  is the image of the basis of  $F_k$  in  $F_k/N$ . A system of basic open sets are sets of the form  $\mathcal{O}_{A,B} = \{N \triangleleft F_k \mid A \subset N, B \cap N = \emptyset\}$  where  $A$  and  $B$  are finite subsets of  $F_k$ . Or the topology can be defined by the metric  $d(N_1, N_2) = 2^{-m}$  where  $m = \max\{n \mid B_{F_k}(n) \cap N_1 = B_{F_k}(n) \cap N_2\}$ . It is easy to see that the topology defined in this way agree with the definition given in the previous paragraph (see [CG05] for a survey of alternative definitions).

Let  $A_\omega = \{a, b_\omega, c_\omega, d_\omega\}$  so that  $\mathcal{F}_2 = \{(G_\omega, A_\omega) \mid \omega \in \Omega_2\}$  is a subset of  $\mathcal{M}_4$ .  $\mathcal{F}_2$  is not closed in  $\mathcal{M}_4$  (see [Gri84]). Given  $\omega \in \Omega_2$ , let  $\{\omega^{(n)}\} \subset \Omega_2 \setminus \Omega_{2,0}$  be a sequence converging to  $\omega$ . It was shown in [Gri84] that the sequence  $\{(G_{\omega^{(n)}}, A_{\omega^{(n)}})\}$  converges in  $\mathcal{M}_4$  to a marked group  $(\tilde{G}_\omega, \tilde{A}_\omega)$  that depends only on  $\omega$ . Moreover,  $(\tilde{G}_\omega, \tilde{A}_\omega) = (G_\omega, A_\omega)$  if and only if  $\omega \in \Omega_2 \setminus \Omega_{2,0}$ . By construction, the group  $\tilde{G}_\omega$  acts naturally on the binary rooted tree for any  $\omega \in \Omega_2$ . However the action is not faithful when  $\omega \in \Omega_{2,0}$ . The modified family  $\{(\tilde{G}_\omega, \tilde{A}_\omega) \mid \omega \in \Omega_2\}$  is a compact subset of  $\mathcal{M}_4$  homeomorphic to  $\Omega_2$  via the map  $\tilde{G}_\omega \mapsto \omega$ .

Observe that a similar procedure can be applied to the family  $\mathcal{G}_2 = \{(G_\omega, S_\omega) \mid \omega \in \Omega_2\}$  to obtain a closed subset  $\{(\tilde{G}_\omega, \tilde{S}_\omega) \mid \omega \in \Omega_2\}$  in  $\mathcal{M}_3$ . In what follows we will mostly be concerned with the modified groups. Therefore, we use notation  $\mathcal{F}_2$  and  $\mathcal{G}_2$  for the modified families and also drop all tildes.



**Growth functions of groups.** Given a group  $G$  and a finite generating set  $S$  of  $G$ , the growth function of  $G$  with respect to  $S$  is defined as  $\gamma_G^S(n) = |B(n)|$  where  $B(n)$  is the ball of radius  $n$  around the identity in the Cayley graph of  $G$  with respect to the generating set  $S$ .

Given two increasing functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$ , write  $f \preceq g$  if there exists a constant  $C > 0$  such that  $f(n) \leq g(Cn)$  for all  $n \in \mathbb{N}$ . Also let  $f \sim g$  mean that  $f \preceq g$  and  $g \preceq f$  with the convention that  $f \prec g$  meaning  $f \preceq g$  but  $f \not\sim g$ . It can be easily observed that  $\sim$  is an equivalence relation and the growth functions of a group with respect to different generating sets are  $\sim$  equivalent. Therefore one can speak of the growth of a group meaning the  $\sim$  equivalence class of its growth functions. Note that if two groups  $(G, S), (H, T) \in \mathcal{M}_k$  are of distance  $2^{-m}$ , then  $\gamma_G^S(n) = \gamma_H^T(n)$  for  $n \leq m$ .

If  $G$  is an *infinite* group and  $H$  a subgroup of finite index, then the growth functions of  $G$  and  $H$  are  $\sim$  equivalent by [Gri84, Proposition 3.1] (note that this is not true if  $G$  is a finite group). Therefore if two finitely generated infinite groups  $G_1$  and  $G_2$  are commensurable (i.e., have finite index subgroups  $H_1, H_2$  which are isomorphic) then their growth functions are  $\sim$  equivalent.

There are three types of growth for groups: If  $\gamma_G \preceq n^d$  for some  $d \geq 0$  then  $G$  is said to be of polynomial growth, if  $\gamma_G \sim e^n$  then it is said to have exponential growth. If neither of this happens then the group is said to have *intermediate growth*. Also the condition  $\gamma_G \prec e^n$  means that  $G$  has *subexponential* growth.

**Definition.** Let  $G$  be a finitely generated group with growth function  $\gamma_G$  corresponding to some generating set. Let  $\gamma_1, \gamma_2$  be two functions such that  $\gamma_1(n) \prec \gamma_2(n) \prec e^n$ .  $G$  is said to have *oscillating growth of type*  $(\gamma_1, \gamma_2)$  if  $\gamma_1 \not\preceq \gamma_G$  and  $\gamma_G \not\preceq \gamma_2$  (i.e., neither  $\gamma_1 \preceq \gamma_G$  nor  $\gamma_G \preceq \gamma_2$ ).

Equivalently, the group  $G$  has oscillating growth of type  $(\gamma_1, \gamma_2)$  if for some (and hence for all) generating set  $S$  we have the following: For every  $C \in \mathbb{N}$  there exists  $m = m(C)$  such that  $\gamma_G^S(Cm) < \gamma_1(m)$  and for every  $D \in \mathbb{N}$  there exists  $k = k(D)$  such that  $\gamma_2(Dk) < \gamma_G^S(k)$ .

Regarding the growth of the groups  $\mathcal{G}_p$  the following are known (recall that  $\gamma_\omega(n)$  denotes the growth function of  $G_\omega$  and when  $\omega \in \Omega_{p,0}$ ,  $G_\omega$  denotes the limit group obtained by the procedure described above).

**Theorem 4.**

- (1) If  $\omega \in \Omega_2 \setminus \Omega_{2,0}$  or  $\omega \in \Omega_{p,\infty}$  if  $p \geq 3$ , then  $G_\omega$  is of intermediate growth.
- (2) If  $\omega \in \Omega_{2,0}$  then  $G_\omega$  is of exponential growth.
- (3) For every  $\omega \in \Omega_2$  or  $\omega \in \Omega_{p,\infty}$ ,  $p \geq 3$ , we have  $e^{\sqrt{n}} \preceq \gamma_\omega(n)$ .
- (4) If there exists a number  $r$  such that every subword of  $\omega$  of length  $r$  contains all the symbols  $\{0, 1, \dots, p\}$  then  $\gamma_\omega(n) \preceq e^{n^\alpha}$  for some  $0 < \alpha < 1$  depending only on  $r$ .
- (5) There is a subset  $\Lambda \subset \Omega_2$  of the cardinality of continuum such that the functions  $\{\gamma_\omega(n) \mid \omega \in \Lambda\}$  are incomparable with respect to  $\preceq$ .

- (6) For any function  $f(n)$  such that  $f(n) \prec e^n$ , there exists  $\omega \in \Omega_2 \setminus \Omega_{2,0}$  for which  $\gamma_\omega(n) \not\preceq f(n)$ .
- (7) If  $\omega = (012)^\infty \in \Omega_2$  is the periodic sequence with period 012 then  $e^{n^{\alpha_0}} \preceq \gamma_\omega(n) \preceq e^{n^{\theta_0}}$ , where  $\alpha_0 = 0.5157$ ,  $\theta_0 = \log(2)/\log(2/x_0)$  and  $x_0$  is the real root of the polynomial  $x^3 + x^2 + x - 2$  ( $\theta_0 \approx 0.7674$ ).
- (8) If  $\omega = (01)^\infty \in \Omega_2$  is periodic with period 01 then  $\exp\left(\frac{n}{\log^{2+\epsilon} n}\right) \preceq \gamma_\omega(n) \preceq \exp\left(\frac{n}{\log^{1-\epsilon} n}\right)$  for any  $\epsilon > 0$ .

*Proof.*

- (1) See [Gri84, Theorem 3.1] and [Gri85a].
- (2) See [Gri84, Lemma 6.1].
- (3) See [Gri84, Theorem 3.2] and [Gri85a, Theorem 4.4] where the lower bound  $e^{\sqrt{n}}$  is proven for a certain subset of  $\Omega_2$  and for  $\Omega_{p,\infty}$ ,  $p \geq 3$ . As all groups mentioned are residually finite  $p$ -groups for some prime  $p$  and are not virtually nilpotent (which can be shown in various ways, for example using the fact that the groups are periodic), the lower bound  $e^{\sqrt{n}}$  follows from a general result of [Gri89].
- (4) See [Gri84] and [BŠ01], [MP01] for explicit upper bounds depending on  $r$ .
- (5) See [Gri84, Theorem 7.2].
- (6) See [Gri84, Theorem 7.1].
- (7) See [Bar01] for the lower bound which improved upon [Leo01]. See [Bar98] for the upper bound.
- (8) See [Ers04].

□

#### 4. PROOF OF THEOREM 1

This section is devoted to the proof of Theorems 1 and 2. We prove these theorems in the case  $p = 2$ . The proof in the case  $p \geq 3$  is completely analogous. To simplify notation, we set  $\Omega = \Omega_2$  and  $\Omega_\infty = \Omega_{2,\infty}$  for the rest of this section.

For any element  $g$  of a group  $G_\omega$ ,  $\omega \in \Omega$ , we denote by  $|g|$  its length relative to the generating set  $A_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ . If  $|g| = n$ , then  $g$  can be expanded into a product  $s_1 s_2 \dots s_n$ , called a geodesic representation, where each  $s_i \in A_\omega$ . For every generator  $s \in A_\omega$  we denote by  $|g|_s$  the number of times this generator occurs in the sequence  $s_1, s_2, \dots, s_n$ . Note that the element  $g$  may admit several geodesic representations and  $|g|_s$  may depend on a representation (for example,  $b_\omega a d_\omega a b_\omega = c_\omega a d_\omega a c_\omega$  for any  $\omega$  starting with 0). Lemmas 1 and 4 below hold for any possible value of the corresponding number  $|g|_s$ .

**Lemma 1.**  $(|g| - 1)/2 \leq |g|_a \leq (|g| + 1)/2$  for all  $g \in G_\omega$ .

*Proof.* It follows from relations (1) that any geodesic representation of an element  $g \in G_\omega$  is of the form

$$g = (s_1)as_2a \dots a(s_k),$$

where each  $s_i \in \{b_\omega, c_\omega, d_\omega\}$  and parentheses indicate optional factors. The lemma follows.  $\square$

**Lemma 2.** *For any word  $w \in \{0, 1\}^*$  of length  $q$  we have  $|g_w| \leq 2^{-q}|g| + 1 - 2^{-q}$ .*

*Proof.* First consider the case when  $w$  is 0 or 1. Let  $g = s_1s_2 \dots s_n$  be a geodesic representation, where each  $s_i \in A_\omega$ . It follows by induction from equation (2) that  $g_w = s_1|_{w_1}s_2|_{w_2} \dots s_n|_{w_n}$ , where  $w_n = w$  and  $w_i = (s_{i+1} \dots s_n)(w)$ , for  $1 \leq i \leq n-1$ . Notice that each section  $s_i|_{w_i}$  is a generator of the group  $G_{\tau(w)}$  or  $e$ . Moreover,  $s_i|_{w_i} = e$  if  $s_i = a$ . Therefore  $|g_w| \leq |g| - |g|_a$ . By Lemma 1,  $|g|_a \geq (|g| - 1)/2$ . Hence  $|g_w| \leq (|g| + 1)/2$ . Equivalently,  $|g_w| - 1 \leq 2^{-1}(|g| - 1)$ .

Now it follows by induction on  $|w|$  that  $|g_w| - 1 \leq 2^{-q}(|g| - 1)$  for any word  $w$  of length  $q$ .  $\square$

For any element  $g \in G_\omega$  and any integer  $q \geq 0$ , let

$$L_q(g) = \sum_{|w|=q} |g_w|$$

**Lemma 3.**  *$L_q(gh) \leq L_q(g) + L_q(h)$  for all  $g, h \in G_\omega$ .*

*Proof.* Since  $(gh)_w = g_{h(w)}h_w$  for any word  $w \in \{0, 1\}^*$ , it follows that  $|(gh)_w| \leq |g_{h(w)}| + |h_w|$ . Summing this inequality over all words  $w$  of length  $q$  and using the fact that  $h$  acts bijectively on such words, we obtain  $L_q(gh) \leq L_q(g) + L_q(h)$ .  $\square$

**Lemma 4.**  *$L_q(g) \leq |g| + 1 - |g|_{h_q}$  for any  $q \geq 1$ , where  $h_q = b_\omega, c_\omega$ , or  $d_\omega$  if the  $q$ -th letter of  $\omega$  is 2, 1, or 0, respectively.*

*Proof.* Let  $n = |g|$ . Consider an arbitrary geodesic representation  $g = s_1s_2 \dots s_n$ , where each  $s_i \in A_\omega$ . It follows by induction from Lemma 3 that  $L_q(g) \leq L_q(s_1) + L_q(s_2) + \dots + L_q(s_n)$ . Fix an arbitrary word  $w \in \{0, 1\}^*$  of length  $q$ . Clearly,  $a|_w = 1$ . Further,  $h_q|_w = 1$  unless  $w = 1 \dots 1$  (in which case  $h_q|_w \in A_{\tau^q\omega} \setminus \{a\}$ ). If  $s$  is any of the other two generators in  $A_\omega$ , then  $s|_w = 1$  unless  $w = 1 \dots 1$  (in which case  $s|_w \in A_{\tau^q\omega} \setminus \{a\}$ ) or  $w = 1 \dots 10$  (in which case  $s|_w = a$ ). Therefore  $L_q(a) = 0$ ,  $L_q(h_q) = 1$ , and  $L_q(s) = 2$  if  $s \in A_\omega$  is neither  $a$  nor  $h_q$ . It follows that  $L_q(g) \leq 2(|g| - |g|_a) - |g|_{h_q}$ . By Lemma 1,  $|g|_a \geq (|g| - 1)/2$ . Hence  $2(|g| - |g|_a) \leq |g| + 1$ .  $\square$

**Lemma 5.** *Suppose that the beginning of length  $q$  of the sequence  $\omega$  contains each of the letters 0, 1, and 2. Then*

$$L_q(g) \leq \frac{5}{6}|g| + \frac{7}{6} + 2^{q-1}$$

for all  $g \in G_\omega$ .

*Proof.* We have  $|g| = |g|_a + |g|_{b_\omega} + |g|_{c_\omega} + |g|_{d_\omega}$  whenever all numbers in the right-hand side are computed for the same geodesic representation of  $g$ . By Lemma 1,  $|g|_a \leq (|g| + 1)/2$ . It follows that  $|g|_s \geq (|g| - 1)/6$  for some  $s \in \{b_\omega, c_\omega, d_\omega\}$ . Lemma 4 implies that  $L_{q_0}(g) \leq |g| + 1 - |g|_s \leq \frac{5}{6}|g| + \frac{7}{6}$  for some  $1 \leq q_0 \leq q$ . In the case  $q_0 = q$ , we are done. Otherwise we notice that  $L_q(g) = \sum_{|w|=q_0} L_{q-q_0}(g_w)$ . By Lemma 4,  $L_{q-q_0}(g_w) \leq |g_w| + 1$  for any word  $w$ . Therefore  $L_q(g) \leq L_{q_0}(g) + 2^{q_0} \leq \frac{5}{6}|g| + \frac{7}{6} + 2^{q-1}$ .  $\square$

Note that the growth function of a group is *sub-multiplicative*, that is,  $\gamma(n+m) \leq \gamma(n)\gamma(m)$  for every  $n, m \in \mathbb{N}$ . It is convenient to extend the argument of a growth function to non-integer values: Given increasing  $f : \mathbb{N} \rightarrow \mathbb{N}$ , define  $\tilde{f} : \mathbb{R}^+ \rightarrow \mathbb{N}$  by  $\tilde{f}(x) = f(\lceil x \rceil)$  for all  $x$  where  $\lceil x \rceil$  is the least natural number bigger than or equal to  $x$ . Observe that  $\tilde{f}(x + \kappa) \leq \tilde{f}(x)$  whenever  $\kappa < 1$ . If  $f : \mathbb{N} \rightarrow \mathbb{N}$  is sub-multiplicative then it is easy to see that  $\tilde{f}(x + y) \leq \tilde{f}(x)\tilde{f}(y)$  for any  $x, y > 0$ .

For the remainder of this section let  $\rho = \frac{131}{132}$ .

**Lemma 6.** *Suppose that the beginning of length  $q$  of the sequence  $\omega$  features each of the letters 0, 1, and 2. Then*

$$\tilde{\gamma}_\omega(x) \leq 2^{2^{q+1}} \left( \tilde{\gamma}_{\tau^q \omega} \left( \frac{x}{11 \cdot 2^q} \right) \right)^{\rho(11 \cdot 2^q)}$$

for any  $x > 0$ .

*Proof.* Let  $n = \lceil x \rceil$  and consider an arbitrary element  $g \in G_\omega$  of length at most  $n$ . By Lemma 2, we have  $|g_w| \leq 2^{-q}n + 1 - 2^{-q}$  for any word  $w \in \{0, 1\}^*$  of length  $q$ . We denote by  $W$  the set of all words  $w$  of length  $q$  such that  $|g_w| > \frac{12}{11} \cdot 2^{-q}(\frac{5}{6}n + \frac{7}{6} + 2^{q-1})$ . In view of Lemma 5, the cardinality of  $W$  satisfies  $|W| < \frac{11}{12} \cdot 2^q$ .

The element  $g$  is uniquely determined by its sections on words of length  $q$  and its restriction to the  $q$ th level of the binary rooted tree. The number of possible choices for the restriction is at most  $2^{2^q}$ . The number of possible choices for the set  $W$  is also at most  $2^{2^q}$ . Once the set  $W$  is specified, the number of possible choices for a particular section  $g_w$  is at most  $\gamma_{\tau^q \omega}(\frac{12}{11} \cdot 2^{-q}(\frac{5}{6}n + \frac{7}{6} + 2^{q-1}))$  if  $w \notin W$  and at most  $\gamma_{\tau^q \omega}(2^{-q}n + 1 - 2^{-q})$  otherwise. Since  $n < x + 1$ , we have  $2^{-q}n + 1 - 2^{-q} < 2^{-q}x + 1$  so that  $\gamma_{\tau^q \omega}(2^{-q}n + 1 - 2^{-q}) \leq \tilde{\gamma}_{\tau^q \omega}(x/2^q)$ . Besides,  $\frac{12}{11} \cdot 2^{-q}(\frac{5}{6}n + \frac{7}{6} + 2^{q-1}) < \frac{10}{11} 2^{-q}x + 1$  so that  $\gamma_{\tau^q \omega}(\frac{12}{11} \cdot 2^{-q}(\frac{5}{6}n + \frac{7}{6} + 2^{q-1})) \leq \tilde{\gamma}_{\tau^q \omega}(\frac{10}{11}x/2^q)$ . Finally, for a fixed set  $W$  the number of possible choices for all sections of  $g$  is

$$\tilde{\gamma}_{\tau^q \omega} \left( \frac{10x}{11 \cdot 2^q} \right)^{2^q - |W|} \tilde{\gamma}_{\tau^q \omega} \left( \frac{x}{2^q} \right)^{|W|} \leq \tilde{\gamma}_{\tau^q \omega} \left( \frac{x}{11 \cdot 2^q} \right)^{10(2^q - |W|) + 11|W|} \leq \tilde{\gamma}_{\tau^q \omega} \left( \frac{x}{11 \cdot 2^q} \right)^{\frac{131}{132} \cdot 11 \cdot 2^q}.$$

Consequently,

$$(3) \quad \tilde{\gamma}_\omega(x) \leq 2^{2^{q+1}} \tilde{\gamma}_{\tau^q(\omega)} \left( \frac{x}{11 \cdot 2^q} \right)^{\rho(11 \cdot 2^q)}.$$

□

As in Section 2, to every infinite word  $\omega = l_1 l_2 \dots$  in  $\Omega_\infty$  we associate an increasing sequence of integers  $t_i = t_i(\omega)$ ,  $i = 0, 1, 2, \dots$ . The sequence is defined inductively. First we let  $t_0 = 0$ . Then, once some  $t_i$  is defined, we let  $t_{i+1}$  to be the smallest integer such that the finite word  $l_{t_i+1} l_{t_i+2} \dots l_{t_{i+1}}$  features each of the letters 0, 1, and 2. Further, let  $q_i = t_i - t_{i-1}$  for  $i = 1, 2, \dots$ .

**Lemma 7.** *Let  $x_m = 11^m \cdot 2^{t_m}$  for any integer  $m > 0$ . Then  $\gamma_\omega(x_m) \leq 10^{\rho^m x_m}$ .*

*Proof.* For any integer  $m > 0$  let  $\alpha_m = \rho(11 \cdot 2^{q_m})$  and  $\beta_m = 2^{q_m+1}$ . Lemma 6 implies that

$$\tilde{\gamma}_{\tau^{t_{m-1}}(\omega)}(x) \leq 2^{\beta_m} \tilde{\gamma}_{\tau^{t_m}(\omega)} \left( \frac{x}{11 \cdot 2^{q_m}} \right)^{\alpha_m}$$

for any  $x > 0$ . Since  $q_1 + q_2 + \dots + q_m = t_m$ , it follows that for any integer  $m > 0$  and real  $x > 0$ ,

$$\tilde{\gamma}_\omega(x) \leq 2^{S_m} \tilde{\gamma}_{\tau^{t_m}(\omega)} \left( \frac{x}{11^m \cdot 2^{t_m}} \right)^{R_m},$$

where  $R_m = \alpha_1 \dots \alpha_m$  and  $S_m = \beta_1 + \alpha_1 \beta_2 + \dots + \alpha_1 \dots \alpha_{m-1} \beta_m$ . In particular,  $\gamma_\omega(x_m) \leq 2^{S_m} \gamma_{\tau^{t_m}(\omega)}(1)^{R_m} = 2^{S_m} 5^{R_m}$ . Since  $R_m = \rho^m x_m$ , it remains to show that  $S_m \leq R_m$ .

We have  $\alpha_m = \frac{11}{2} \rho \beta_m = \frac{131}{24} \beta_m > 5 \beta_m$ . Notice that  $q_m \geq 3$  so that  $\beta_m \geq 16$ . Hence  $\alpha_m - \beta_m > 64$ . Now the inequality  $S_m \leq R_m$  is proved by induction on  $m$ . First of all,  $S_1 = \beta_1 < \alpha_1 = R_1$ . Then, assuming  $S_m \leq R_m$  for some  $m > 0$ , we get  $S_{m+1} = S_m + \alpha_1 \dots \alpha_m \beta_{m+1} \leq R_m + \alpha_1 \dots \alpha_m \beta_{m+1} = \alpha_1 \dots \alpha_m (1 + \beta_{m+1}) < R_{m+1}$ . □

Recall some notation from Section 2 (for brevity, we drop index  $p$ ). For any  $C \geq 0$  let  $\Omega_C$  denote the set of all infinite words  $\omega \in \Omega_\infty$  such that  $t_n(\omega) \leq Cn$  for sufficiently large  $n$ . Given  $\epsilon > 0$ , let  $\Omega_{C,\epsilon}$  denote the set of all  $\omega \in \Omega_C$  such that  $q_{n+1} = t_{n+1}(\omega) - t_n(\omega) \leq \epsilon t_n(\omega)$  for sufficiently large  $n$ .

Now we can prove the next theorem, which is a more detailed version of Theorem 2 (in the case  $p = 2$ ).

**Theorem 5.** *Let  $C > 0$  and  $\alpha > 1 - \frac{\log(\rho^{-1})}{\log(11 \cdot 2^C)}$ . Then there exists  $\epsilon > 0$  such that  $\gamma_\omega(n) \leq e^{n^\alpha}$  for any  $\omega \in \Omega_{C,\epsilon}$ .*

*Proof.* Let  $\kappa = \frac{\log(\rho^{-1})}{\log(11 \cdot 2^C)}$ . Note that  $0 < \kappa < 1$ . Choose  $\epsilon > 0$  small enough so that  $(\epsilon + 1)(1 - \kappa) < \alpha$ . Let  $\omega \in \Omega_{C,\epsilon}$ . Then there exists an integer  $N > 0$  such

that  $t_m \leq Cm$  and  $q_{m+1} = t_{m+1} - t_m \leq \epsilon t_m$  for  $m \geq N$ . By the choice of  $\kappa$  we have

$$\rho^m = \left( \frac{1}{(11 \cdot 2^C)^\kappa} \right)^m = \frac{1}{(11^m \cdot 2^{Cm})^\kappa} \leq \frac{1}{(11^m \cdot 2^{t_m})^\kappa} = x_m^{-\kappa}$$

for any  $m \geq N$ . Since

$$x_{m+1} = 11^{m+1} \cdot 2^{t_{m+1}} = 11 \cdot 2^{q_{m+1}} \cdot 11^m \cdot 2^{t_m} = 11 \cdot 2^{q_{m+1}} \cdot x_m,$$

we obtain

$$\begin{aligned} x_{m+1}^{1-\kappa} &= (11 \cdot 2^{q_{m+1}} \cdot x_m)^{1-\kappa} \leq 11^{1-\kappa} (11^{\epsilon m} \cdot 2^{\epsilon t_m} \cdot x_m)^{1-\kappa} \\ &= 11^{1-\kappa} x_m^{(\epsilon+1)(1-\kappa)} \leq 11^{1-\kappa} x_m^\alpha. \end{aligned}$$

Consider an arbitrary integer  $n \geq x_N$ . We have  $x_m \leq n \leq x_{m+1}$  for some  $m \geq N$ . By Lemma 7,

$$\gamma_\omega(n) \leq \gamma_\omega(x_{m+1}) \leq 10^{\rho^{m+1} x_{m+1}}.$$

By the above,

$$\rho^{m+1} x_{m+1} \leq x_{m+1}^{1-\kappa} \leq 11^{1-\kappa} x_m^\alpha \leq 11^{1-\kappa} n^\alpha,$$

hence

$$\gamma_\omega(n) \leq 10^{11^{1-\kappa} n^\alpha} = D^{n^\alpha},$$

where  $D = 10^{11^{1-\kappa}}$ . Thus  $\gamma_\omega(n) \leq D^{n^\alpha} \sim e^{n^\alpha}$ . □

Suppose  $\mu$  is a Borel probability measure on  $\Omega$  that is invariant and ergodic relative to the shift transformation  $\tau : \Omega \rightarrow \Omega$ . Since  $\Omega_\infty$  is a Borel, shift invariant set, the measure  $\mu$  is either supported on  $\Omega_\infty$  or else  $\mu(\Omega_\infty) = 0$ . Theorem 1 will be derived from Theorem 5 using the following lemma.

**Lemma 8.** *If the measure  $\mu$  is supported on  $\Omega_\infty$ , then there exists  $C_0 > 0$  such that  $t_n(\omega)/n \rightarrow C_0$  as  $n \rightarrow \infty$  for  $\mu$ -almost all  $\omega \in \Omega$ . Consequently,  $\mu(\Omega_{C,\epsilon}) = 1$  for any  $C > C_0$  and  $\epsilon > 0$ . In the case  $\mu$  is the uniform Bernoulli measure on  $\Omega$ , we can take  $C_0 < 7.3$ .*

*Proof.* For any finite word  $w$  over the alphabet  $\{0, 1, 2\}$  let  $T(w)$  denote the maximal number of non overlapping sub-words of  $w$  each containing all the letters. Clearly,  $T(w) \leq |w|/3$ . It is easy to see that  $T(w_1) + T(w_2) \leq T(w_1 w_2) \leq T(w_1) + T(w_2) + 1$  for any words  $w_1$  and  $w_2$ . It follows by induction that  $T(w_0) + T(w_1) + \dots + T(w_k) \leq T(w_0 w_1 \dots w_k) \leq T(w_0) + T(w_1) + \dots + T(w_k) + k$  for any words  $w_0, w_1, \dots, w_k$ .

For any  $\omega \in \Omega$  and integer  $m > 0$  let  $T_m(\omega) = T(\omega_m)$ , where  $\omega_m$  is the beginning of length  $m$  of the sequence  $\omega$ . We are going to show that for  $\mu$ -almost all  $\omega$  there is a limit of  $T_m(\omega)/m$  as  $m \rightarrow \infty$ . Note that  $T_m$  is a bounded ( $0 \leq T_m \leq m/3$ ) Borel function on  $\Omega$ . Let



$$I_m = \int_{\Omega} T_m d\mu.$$

If  $\omega \in \Omega_{\infty}$  then  $T_m(\omega) > 0$  for  $m$  large enough. Since the measure  $\mu$  is supported on the set  $\Omega_{\infty}$ , it follows that  $I_m > 0$  for  $m$  large enough.

Given integers  $m_1, m_2 > 0$ , the beginning of length  $m_1 + m_2$  of any sequence  $\omega \in \Omega$  is represented as the concatenation of two words, the beginning of length  $m_1$  of the same sequence and the beginning of length  $m_2$  of the sequence  $\tau^{m_1}(\omega)$ . Therefore  $T_{m_1+m_2}(\omega) \geq T_{m_1}(\omega) + T_{m_2}(\tau^{m_1}(\omega))$ . Integrating this inequality over  $\Omega$  and using shift-invariance of the measure  $\mu$ , we obtain  $I_{m_1+m_2} \geq I_{m_1} + I_{m_2}$ . Now the standard argument implies that  $I_m/m \rightarrow I$  as  $m \rightarrow \infty$ , where  $I = \sup_{k \geq 1} I_k/k$ . Note that  $0 < I \leq 1/3$ .

Let  $\Omega_{\mu}$  denote the Borel set of all sequences  $\omega \in \Omega$  such that for any integer  $m > 0$  we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} T_m(\tau^i(\omega)) = I_m.$$

Birkhoff's ergodic Theorem implies that  $\Omega_{\mu}$  is a set of full measure:  $\mu(\Omega_{\mu}) = 1$ . Consider an arbitrary  $\omega \in \Omega$  and integers  $m > 0$  and  $k \geq 2m$ . Let  $l = \lfloor k/m \rfloor$ , the integer part of  $k/m$ . For any integer  $j$ ,  $0 \leq j < m$ , we represent the beginning of length  $k$  of  $\omega$  as the concatenation of  $l$  words  $w_0 w_1 \dots w_l$ , where  $w_0$  is of length  $j$ ,  $w_l$  is of length  $k - lm + m - j$ , and the other words are of length  $m$ . By the above,

$$T_k(\omega) - l \leq \sum_{i=0}^l T(w_i) \leq T_k(\omega).$$

By construction,  $T(w_i) = T_m(\tau^{(i-1)m+j}(\omega))$  for  $1 \leq i \leq l-1$ . Besides,  $l \leq k/m$  and  $0 \leq T(w_0) + T(w_l) \leq (k - lm + m)/3 < 2m/3$ . Therefore

$$T_k(\omega) - k/m - 2m/3 \leq \sum_{i=1}^{l-1} T_m(\tau^{(i-1)m+j}(\omega)) \leq T_k(\omega).$$

Summing the latter inequalities over  $j$  ranging from 0 to  $m-1$ , we obtain

$$mT_k(\omega) - k - 2m^2/3 \leq \sum_{i=0}^{(l-1)m-1} T_m(\tau^i(\omega)) \leq mT_k(\omega).$$

Since  $0 \leq \sum_{i=(l-1)m}^{k-1} T_m(\tau^i(\omega)) \leq (k - lm + m)m/3 < 2m^2/3$ , it follows that

$$mT_k(\omega) - k - 2m^2/3 \leq \sum_{i=0}^{k-1} T_m(\tau^i(\omega)) \leq mT_k(\omega) + 2m^2/3.$$

Then

$$\left| \frac{1}{k} T_k(\omega) - \frac{1}{mk} \sum_{i=0}^{k-1} T_m(\tau^i(\omega)) \right| \leq \frac{1}{m} + \frac{2m}{3k}.$$

At this point, let us assume that  $\omega \in \Omega_{\mu}$ . Fixing  $m$  and letting  $k$  go to infinity in the latter estimate, we obtain that all limit points of the sequence  $\{T_k(\omega)/k\}_{k \geq 1}$  lie in the interval  $[I_m/m - 1/m, I_m/m + 1/m]$ . Letting  $m$  go to infinity as well, we obtain that  $T_k(\omega)/k \rightarrow I$  as  $k \rightarrow \infty$ .

Given  $\omega \in \Omega_\infty$ , there is a simple relation between sequences  $\{T_m(\omega)\}_{m \geq 1}$  and  $\{t_n(\omega)\}_{n \geq 1}$ . Namely,  $t_n(\omega) \leq m$  if and only if  $T_m(\omega) \geq n$ . In particular,  $T_{t_n(\omega)}(\omega) = n$ . Since  $T_m(\omega)/m \rightarrow I$  as  $m \rightarrow \infty$  for any  $\omega \in \Omega_\mu$ , it easily follows that  $t_n(\omega)/n \rightarrow C_0$ , where  $C_0 = I^{-1}$ , for any  $\omega$  in  $\Omega_\mu \cap \Omega_\infty$ , a set of full measure.

Now consider the case  $\mu$  is the uniform Bernoulli measure. To estimate the limit  $C_0$  in this case, we are going to evaluate the integral  $I_7$ . For any integer  $k \geq 0$  let  $N_k$  denote the number of words  $w$  of length 7 over the alphabet  $\{0, 1, 2\}$  such that  $T(w) = k$ . Then  $I_7 = 3^{-7} \sum_{k \geq 0} k N_k$ . Since  $N_k = 0$  for  $k > 2$ , we have  $N_0 + N_1 + N_2 = 3^7$  and  $I_7 = (N_1 + 2N_2)/3^7$ . Let us compute the numbers  $N_0$  and  $N_2$ . A word  $w$  of length 7 satisfies  $T(w) = 0$  if it does not use one of the letters. The number of words missing one particular letter is  $2^7$ . Also, there are three words 0000000, 1111111, and 2222222 that miss two letters. It follows that  $N_0 = 3 \cdot 2^7 - 3 = 381$ . To compute  $N_2$ , we represent an arbitrary word  $w$  of length 7 as  $w_1 l w_2$ , where  $w_1$  and  $w_2$  are words of length 3 and  $l$  is a letter. There are two cases when  $T(w) = 2$ . In the first case, each of the words  $w_1$  and  $w_2$  contains all letters, then  $l$  can be arbitrary. In the second case, either  $w_1$  or  $w_2$  misses exactly one letter, then  $l$  must be the missing letter and the other word must contain all letters. It follows that  $N_2 = (3!)^2 \cdot 3 + 2M \cdot 3!$ , where  $M$  is the number of words of length 3 that miss exactly one of the letters 0, 1, and 2. It is easy to observe that  $M = 3^3 - 3! - 3 = 18$ , then  $N_2 = 324$ . Now  $N_1 = 3^7 - N_0 - N_2 = 2187 - 381 - 324 = 1482$ . Finally,  $I_7 = (N_1 + 2N_2)/3^7 = (1482 + 2 \cdot 324)/3^7 = 710/729$ . Now we can estimate the limits. As shown earlier,  $I \geq I_7/7 = 710/(7 \cdot 729) > 100/729$ , then  $C_0 = I^{-1} < 7.3$ . □

Now we are ready to complete the proof of Theorem 1.

Take any  $C > C_0$ , where  $C_0$  is as in Lemma 8. By Theorem 5, there exists  $\epsilon > 0$  and  $0 < \alpha < 1$  such that  $\gamma_\omega(n) \preceq e^{n^\alpha}$  for all  $\omega \in \Omega_{C, \epsilon}$ . The set  $\Omega_{C, \epsilon}$  has full measure by Lemma 8. In the case when  $\mu$  is the uniform Bernoulli measure, we can assume that  $C < 7.3$  by Lemma 8. Consequently, we can choose  $\alpha = 1 - \kappa$ , where  $\kappa = \log \frac{132}{131} / \log(11 \cdot 2^{7.3})$ . One can compute that  $\kappa > 0.001$ . □

## 5. PROOF OF THEOREM 3

Recall that we are in the case  $p = 2$  so that we use the notation  $\Omega = \Omega_2$  and  $\Omega_0 = \Omega_{2,0}$ . Also, recall  $\theta_0$  is as defined before Theorem 3. We begin with preliminary lemmas.

**Lemma 9.** *Let  $g$  be a function of natural argument and let  $\mathcal{L}_g \subset \mathcal{M}_k$  be the subset consisting of marked groups  $(G, S)$  such that  $g \not\preceq \gamma_G^S$ . Then  $\mathcal{L}_g$  is a  $G_\delta$  subset of  $\mathcal{M}_k$  (i.e., a countable intersection of open sets).*

*Proof.* Given  $(G, S) \in \mathcal{L}_g$  and  $C \in \mathbb{N}$ , let  $K = K((G, S), C)$  be such that  $\gamma_G^S(CK) < g(K)$  (such  $K$  exists since  $g \not\preceq \gamma_G^S$ ). Let  $\mathcal{B}_{((G, S), C)}$  denote the ball

of radius  $2^{-CK}$  (in the metric defined in the space of marked groups) centered at  $(G, S)$ . We claim that

$$\mathcal{L}_g = \bigcap_{C \in \mathbb{N}} \bigcup_{(G, S) \in \mathcal{L}_g} \mathcal{B}_{((G, S), C)}.$$

The inclusion  $\subset$  is clear. For the other inclusion, let  $(H, T)$  be an element of the right hand side. Then for any  $C \in \mathbb{N}$  there is  $(G, S) \in \mathcal{L}_g$  such that  $(H, T) \in \mathcal{B}_{((G, S), C)}$ . Therefore for  $K = K((G, S), C)$  we have  $\gamma_H^T(CK) = \gamma_G^S(CK) < g(K)$  and hence  $g \not\preceq \gamma_H^T$ , which shows that  $(H, T) \in \mathcal{L}_g$ .  $\square$

**Lemma 10.** *Let  $f$  be a function of natural argument and let  $\mathcal{U}_f \subset \mathcal{M}_k$  be the subset consisting of marked groups  $(G, S)$  such that  $\gamma_G^S \not\preceq f$ . Then  $\mathcal{U}_f$  is a  $G_\delta$  subset of  $\mathcal{M}_k$ .*

*Proof.* The proof is analogous to the proof of Lemma 9.  $\square$

Now we are going to prove each part of Theorem 3.

**Part a)** Suppose we are given  $\theta > \theta_0$  and a function  $f(n) \prec e^n$ . Let  $\eta = (012)^\infty \in \Omega$  and recall that we have  $\gamma_\eta(n) \preceq e^{n^{\theta_0}}$  (Theorem 4, part 7). Consider the set  $\mathcal{X} = \{(G_\omega, S_\omega) \mid \tau^k(\omega) = \eta \text{ for some } k\} \subset \mathcal{G}_2$ . Since  $\mathcal{G}_2$  is homeomorphic to  $\Omega$  via  $(G_\omega, S_\omega) \mapsto \omega$ , the set  $\mathcal{X}$  is dense in  $\mathcal{G}_2$ . For any  $\omega \in \Omega \setminus \Omega_0$ , the groups  $G_\omega$  and  $G_{\tau(\omega)} \times G_{\tau(\omega)}$  are commensurable by [Gri84, Theorem 2.2]. Therefore we have

$$\gamma_\omega \sim \gamma_{\tau(\omega)}^2$$

and for any  $\omega$ ,  $\tau^k(\omega) = \eta$  it follows that

$$\gamma_\omega \sim \gamma_\eta^{2^k} \preceq (e^{n^{\theta_0}})^{2^k} \sim e^{n^{\theta_0}}.$$

Let  $g(n) = e^{n^\theta}$  so that  $\mathcal{X} \subset \mathcal{L}_g$ , where  $\mathcal{L}_g$  is defined in Lemma 9.

According to Theorem 4, part 2, the set  $\mathcal{Y} = \{(G_\omega, S_\omega) \mid \omega \in \Omega_0\}$ , which is dense in  $\mathcal{G}_2$ , consists of groups of exponential growth. In particular,  $\mathcal{Y} \subset \mathcal{U}_f$ , where  $\mathcal{U}_f \subset \mathcal{M}_3$  is defined in Lemma 10. By Lemmas 9 and 10, the sets  $\mathcal{L}_g \cap \mathcal{G}_2$  and  $\mathcal{U}_f \cap \mathcal{G}_2$  are dense  $G_\delta$  subsets of  $\mathcal{G}_2$ . Since  $\mathcal{G}_2$  is compact, their intersection is also a dense  $G_\delta$  subset of  $\mathcal{G}_2$ . For any  $(G, S) \in \mathcal{L}_g \cap \mathcal{U}_f \cap \mathcal{G}_2$ , we have  $g \not\preceq \gamma_G^S$  and  $\gamma_G^S \not\preceq f$ .

**Part b)** This part is a corollary of part a) with  $f(n) = e^{\frac{n}{\log n}}$  as  $e^{n^\beta} \prec f(n)$  for any  $\beta < 1$ .

**Part c)** The proof of this part is analogous to part a). Let us denote by  $\Omega' = \{0, 1\}^\mathbb{N} \subset \Omega$ . We set  $\zeta = (01)^\infty$ ,  $\mathcal{X} = \{(G_\omega, S_\omega) \mid \tau^k(\omega) = \zeta \text{ for some } k\}$  and  $\mathcal{Y} = \{(G_\omega, S_\omega) \mid \omega \in \Omega_0 \cap \Omega'\}$ . According to Theorem 4, part 8, for any  $\epsilon > 0$  the function  $g(n) = \exp\left(\frac{n}{\log^{1-\epsilon} n}\right)$  grows faster than  $\gamma_\zeta$ . Since  $g \sim g^2$ , it follows that  $\gamma_\omega \preceq g$  whenever  $(G_\omega, S_\omega) \in \mathcal{X}$ . It remains to apply Lemmas 9 and 10.

## 6. PROOF OF THEOREMS 1' AND 3'

As it was mentioned in the introduction there is a natural embedding  $\iota_k : \mathcal{M}_k \rightarrow \mathcal{M}_{k+1}$  given by  $\iota_k((G, A)) = (G, A')$  where  $A' = \{a_1, \dots, a_k, a_{k+1}\}$  if  $A = \{a_1, \dots, a_k\}$  and  $a_{k+1} = 1$  in  $G$ . This induces an embedding  $\iota_{k,n} : \mathcal{M}_k \rightarrow \mathcal{M}_{k+n}$  for all  $k, n$  and given a subset  $X \subset \mathcal{M}_k$  one can consider its homeomorphic image  $\iota_{k,n}(X) \subset \mathcal{M}_{k+n}$ .

There are two natural ways of replacing one generating set  $A$  of a group  $G$  by another. The one, as was just suggested, by adding one more formal generator representing the identity (and placing it for definiteness at the end), or applying to a generating set *Nielsen transformations* which are given by (see [MKS66]):

- i) Exchanging two generators ,
- ii) Replacing a generator  $a \in A$  by its inverse  $a^{-1}$ ,
- iii) Replacing  $a_i \in A$  by  $a_i a_j$  where  $a_i \neq a_j$ .

Note that these transform generating sets into generating sets. It is in general incorrect that two generating sets of size  $k$  of a group are related by a sequence of Nielsen transformations (i.e. by an automorphism of the free group  $F_k$ ), but we have the following:

**Proposition 1.** *Let  $(G, A) \in \mathcal{M}_k$  and  $(G, B) \in \mathcal{M}_n$ . Let  $\iota_{k,n}(G, A) = (G, A')$  and  $\iota_{n,k}(G, B) = (G, B')$ . so that  $(G, A'), (G, B') \in \mathcal{M}_{n+k}$ . Then  $A'$  can be transformed into  $B'$  by a sequence of Nielsen transformations.*

*Proof.* Let  $A' = \{a_1, \dots, a_k, a_{k+1}, \dots, a_{k+n}\}$  and  $B' = \{b_1, \dots, b_n, b_{n+1}, \dots, b_{n+k}\}$  where  $a_{k+1} = \dots = a_{k+n} = b_{n+1}, \dots = b_{n+k} = 1$ . For all  $b_i$ ,  $1 \leq i \leq n$  there is a word  $B_i \in \{a_1, \dots, a_k\}^\pm$  such that  $b_i = B_i$ . By a sequence of Nielsen transformations (acting trivially on  $a_i$ ,  $i \leq k$ ) we can transform  $A'$  into  $A'' = \{a_1, \dots, a_k, B_1, \dots, B_n\} = \{a_1, \dots, a_k, b_1, \dots, b_n\}$ . In a similar way  $B'$  can be transformed into  $B'' = \{b_1, \dots, b_n, a_1, \dots, a_k\}$  by a sequence of Nielsen transformations. It is clear that  $B''$  can be obtained from  $A''$  by permuting the generators, which can be achieved by a sequence of Nielsen transformations.  $\square$

Taking the inductive limit  $\mathcal{M} = \lim_{\rightarrow} \mathcal{M}_k$  and setting  $A_\infty = \{a_1, a_2, \dots\}$  the previous proposition shows (as observed by Champetier in [Cha00]) that the group of Nielsen transformations over an infinite alphabet (that is, the group  $\text{Aut}_{fin}(F_\infty)$  of finitary automorphisms of a free group  $F_\infty$  of countably infinite rank) acts on  $\mathcal{M}$  in such a way that if two pairs  $(G, A), (G, B) \in \mathcal{M}$  represent the same group then they belong to the same orbit of the action of  $\text{Aut}_{fin}(F_\infty)$  on  $\mathcal{M}$  (and it is clear the points in the orbit all represent the same group). In [Cha00] it was shown that this action, which is by homeomorphisms and hence is Borel, is not *tame* (in other terminology not measurable, or not smooth). As was mentioned in the introduction, the question of existence of  $\text{Aut}_{fin}(F_\infty)$ -invariant (or at least quasi-invariant) measure is important for the topic of random groups.

There are more general ways of embedding  $\mathcal{M}_k$  into  $\mathcal{M}_l$ . Assume we have a subset  $X = \{(G_i, A_i) \mid i \in I\} \subset \mathcal{M}_k$  where  $A_i = \{a_1^{(i)}, \dots, a_k^{(i)}\}$ . Let  $F_k$  be a free

group on  $\{a_1, \dots, a_k\}$  and suppose that there are words  $B_j(a_1, \dots, a_k) \in F_k$  for  $1 \leq j \leq m$  such that for all  $i \in I$ , the set

$$B_i = \{B_1(a_1^{(i)}, \dots, a_k^{(i)}), \dots, B_m(a_1^{(i)}, \dots, a_k^{(i)})\}$$

is a generating set for  $G_i$ . Let  $Y = \{(G_i, B_i) \mid i \in I\} \subset \mathcal{M}_m$ .

**Proposition 2.** *The map  $\varphi : X \rightarrow Y$  given by  $\varphi((G_i, A_i)) = (G_i, B_i)$  is a homeomorphism.*

*Proof.* Let  $X' = \iota_{k+m}(X)$  and  $Y' = \iota_{m+k}(Y)$ . By the previous proposition, there is an automorphism of  $F_{k+m}$  (realized by a sequence of Nielsen transformations) which induces a homeomorphism  $\phi$  of  $\mathcal{M}_{k+m}$  which maps  $X'$  onto  $Y'$ . It is clear that  $\varphi$  is the restriction of  $\iota_{m+k}^{-1} \circ \phi \circ \iota_{k+m}$  to  $X$ .  $\square$

We are ready to prove the theorems.

Theorems 1 and 3 show that theorems 1' and 3' hold for  $k = 4$ . Using the previous propositions it immediately follows that they hold for values  $k \geq 4$ .

For  $k = 3$  observe that by virtue of equations (1), for every  $\omega \in \Omega$  we have  $d_\omega = b_\omega c_\omega$  and hence the groups  $G_\omega$  are generated by  $\{a, b_\omega, c_\omega\}$ . Therefore by proposition 2 we obtain the result for  $k = 3$ .

The case  $k = 2$  is more delicate. There are several methods of embedding a group into a 2-generated group. We need an embedding that preserves the property to have intermediate growth. To accomplish this we use an idea from [Gri84]. Let  $T$  be the rooted tree with branch index 4, 2, 2, 2,  $\dots$ . Given  $\omega \in \Omega$ , let  $e$  be the automorphism of  $T$  which cyclically permutes the first level vertices and let  $f_\omega$  be the automorphism given by  $(b_\omega, c_\omega, a, 1)$ . Set  $M_\omega = \langle x, y_\omega \rangle$ . This gives an embedding

$$\begin{array}{rcl} \psi : M_\omega & \rightarrow & S_4 \ltimes G_\omega^4 \\ x & \mapsto & \sigma \quad (1, 1, 1, 1) \\ y_\omega & \mapsto & (b_\omega, c_\omega, a, 1) \end{array}$$

where  $\sigma$  is the cyclic permutation of order 4 in  $S_4$ . Let

$$\bar{M}_\omega = \langle y_\omega, xy_\omega x^{-1}, x^2 y_\omega x^{-2}, x^3 y_\omega x^{-3} \rangle$$

and observe that  $\bar{M}_\omega$  has index 4 in  $M_\omega$ . The equalities

$$\begin{array}{rcl} \psi(y_\omega) & = & (b_\omega, c_\omega, a, 1) \\ \psi(xy_\omega x^{-1}) & = & (c_\omega, a, 1, b_\omega) \\ \psi(x^2 y_\omega x^{-2}) & = & (a, 1, b_\omega, c_\omega) \\ \psi(x^3 y_\omega x^{-3}) & = & (1, b_\omega, c_\omega, a) \end{array}$$

show that  $\bar{M}_\omega$  is a sub-direct product of  $G_\omega^4$ . Hence if  $G_\omega$  has intermediate growth so does  $M_\omega$  and if the growth of  $G_\omega$  is bounded above by a function of the form  $e^{n^\alpha}$  then the same holds for the growth function of  $M_\omega$ . Similarly, if  $G_\omega$  has oscillating growth of type  $(e^{n^\theta}, f)$ , so does  $M_\omega$ .

One can observe that the branch algorithm solving the word problem for groups  $G_\omega$  (described in [Gri84]) can be adapted to the groups  $M_\omega$ : The covering group will be  $\mathbb{Z}_4 * \mathbb{Z}_2$ , given a word  $g$  in the normal form in  $\mathbb{Z}_4 * \mathbb{Z}_2$  first one checks whether the exponent of  $e$  in  $g$  is divisible by 4 or not. If not then the element  $g$  does not belong to the first level stabilizer and hence  $g \neq 1$ . Otherwise one computes the sections of  $g$  and then applies the classical branch algorithm to the sections of  $g$  with oracle  $\omega$ . This shows that for two sequences  $\omega, \eta \in \Omega \setminus \Omega_0$  which have common prefix of length  $n$ , the Cayley graphs of the groups  $M_\omega$  and  $M_\eta$  will have isomorphic balls of radius  $2^{n-1}$ . Therefore we consider the subset  $X = \{(M_\omega, L_\omega) \mid \omega \in \Omega \setminus \Omega_0\} \subset \mathcal{M}_2$  where  $L_\omega = \{x, y_\omega\}$  and take its closure in  $\mathcal{M}_2$  to obtain a Cantor set in  $\mathcal{M}_2$ . The new limit groups  $M_\omega$  for  $\omega \in \Omega_0$  will have a finite index subgroup which is a sub-direct product in the group  $G_\omega^4$ , and therefore are of exponential growth. Thus the limit groups  $M_\omega$ ,  $\omega \in \Omega_0$  will have exponential growth and therefore similar arguments used to prove Theorem 3 can be applied in this case too. Also note that when  $\omega \in \Omega_\infty$  then  $M_\omega$  and  $G_\omega^4$  are abstractly commensurable i.e. have finite index subgroups which are isomorphic.

For  $p \geq 3$  a similar construction can be done by setting  $M_\omega = \langle x, y_\omega \rangle$  as the group of automorphisms of the tree with branch index  $p^2, p, p, \dots$ , where  $x$  is the cyclic permutation of order  $p$  and  $y_\omega = (b_\omega, c_\omega, a, 1, \dots, 1)$ . One can observe that  $M_\omega$  in this case is a sub-direct product in  $G_\omega^{p^2}$  and  $M_\omega$  is abstractly commensurable with  $G_{\tau(\omega)}^{p^2}$  when  $\omega \in \Omega_{p,\infty}$ . This allows to prove the theorem in the case  $p \geq 3$ .

## 7. CONCLUDING REMARKS

Let  $G_\omega^{um}$ ,  $\omega \in \Omega_{p,0}$  denote the unmodified groups as defined in Section 3 (i.e., the groups before modifying countably many groups corresponding to eventually constant sequences). Note that for fixed prime  $p$  we have  $G_{0^\infty}^{um} = G_{1^\infty}^{um} = \dots = G_{p^\infty}^{um}$  as subgroups of the  $p$ -ary rooted tree. The limit groups  $G_\omega$ ,  $\omega \in \Omega_{p,0}$  map onto the corresponding group  $G_\omega^{um}$ . When  $p = 2$  and  $\omega \in \Omega_2$  is a constant sequence then  $G_\omega^{um}$  is isomorphic to the infinite dihedral group [Gri84, Lemma 2.1] and hence has linear growth. This shows that  $G_\omega^{um}$  has polynomial growth for  $\omega \in \Omega_{2,0}$ . For  $p \geq 3$  and  $\omega \in \Omega_p$  a constant sequence, the groups  $G_\omega^{um}$  were considered in [Gri00] and were shown to be regular branch self-similar groups. As these groups are residually finite  $p$ -groups, the main result of [Gri89] shows that for all such groups  $e^{\sqrt{n}}$  is a lower bound for their growth functions. Therefore for all primes  $p$  and  $\omega \in \Omega_{p,0}$ , the groups  $G_\omega$  have super-polynomial growth. As mentioned in Theorem 4, for  $p = 2$  the groups  $G_\omega$   $\omega \in \Omega_{2,0}$  are known to have exponential growth. An extension of this fact to  $p > 2$  would generalize Theorem 3 to all primes  $p$ . For  $p = 3$  and  $\omega \in \Omega_p$  a constant sequence, the group  $G_\omega^{um}$  coincides with the Fabrykowski-Gupta group studied in [FG91]. In [BP09] it was shown that the growth of this group satisfies

$$e^{n^{\frac{\log 3}{\log 6}}} \preceq \gamma(n) \preceq e^{\frac{n(\log \log n)^2}{\log n}}.$$



A more general problem is the following: Given two increasing functions  $\gamma_1, \gamma_2$  such that  $\gamma_i(n) \sim \gamma_i(n)^p$ ,  $i = 1, 2$  and  $\gamma_1(n) \prec \gamma_2(n)$ , consider the set

$$\mathcal{W}_{\gamma_1, \gamma_2} = \{\omega \in \Omega_p \mid \gamma_1(n) \preceq \gamma_\omega(n) \preceq \gamma_2(n)\}.$$

As mentioned before, for  $\omega \in \Omega_p \setminus \Omega_{p,0}$  the groups  $G_\omega$  and  $G_{\tau(\omega)}$  are commensurable and hence  $\gamma_\omega \sim \gamma_{\tau(\omega)}^p$ . This shows that  $\mathcal{W}_{\gamma_1, \gamma_2}$  is  $\tau$  invariant, and hence for any  $\tau$  invariant ergodic measure  $\mu$  defined on  $\Omega_p$  we have  $\mu(\mathcal{W}_{\gamma_1, \gamma_2}) = 0$  or  $1$ . A natural direction for investigation would be to determine functions  $\gamma_1, \gamma_2$  for which the set  $\mathcal{W}_{\gamma_1, \gamma_2}$  has full measure (and make  $\gamma_1, \gamma_2$  as close to each other as possible while keeping  $\mu(\mathcal{W}_{\gamma_1, \gamma_2}) = 1$ ). Theorem 1, part (b) together with Theorem 4, part (3) can be interpreted as  $\mu(\mathcal{W}_{\gamma_1, \gamma_2}) = 1$ , where  $\gamma_1(n) = e^{n^{0.5}}$ ,  $\gamma_2(n) = e^{n^{0.999}}$  and  $\mu$  is the uniform Bernoulli measure on  $\Omega_2$ .

The idea of statements similar to Lemmas 9 and 10, which descends to the paper of A. Stepin [Ste96], is based on the fact that many group properties are formulated in "local terms" with respect to the topology on  $\mathcal{M}_k$ . This includes properties such as to be amenable, to be LEK (locally embeddable into the class  $K$  of groups), to be sofic, to be hyperfinite, etc. (see, e.g., [CSC10]).

In all these and other cases one can state that for any  $k$  the subset  $X_{\mathcal{P}} \subset \mathcal{M}_k$  of groups satisfying a local property  $\mathcal{P}$  is a  $G_\delta$  set in  $\mathcal{M}_k$ . So if a subset  $Y \subset \mathcal{M}_k$  has a dense subset of groups satisfying property  $\mathcal{P}$  then it contains dense  $G_\delta$  subset satisfying property  $\mathcal{P}$ . In some cases like LEF (locally embeddable into finite groups), LEA (locally embeddable into amenable groups), sofic and hyperfinite groups, the corresponding set is a closed subset in  $\mathcal{M}_k$  and there is not a big outcome of the above argument. But for properties such as to be amenable, to have particular type of growth and some other properties, the above observation gives a nontrivial information about the structure of the space of groups.

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MUSTAFA G. BENLI

*E-mail address:* mbenli@math.tamu.edu

ROSTISLAV GRIGORCHUK

*E-mail address:* grigorch@math.tamu.edu

YAROSLAV VOROBETS

*E-mail address:* yvorobet@math.tamu.edu

TEXAS A&M UNIVERSITY, MAILSTOP 3368, COLLEGE STATION, TX 77843–3368, USA.